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Finite deformation analysis of a thin-walled tube sliding on a rough rigid rod

A.D. KYDONIEFS[1] and A.J.M. SPENCER[2,*]

[1]*Department of Mathematics and Physics, School of Technology, University of Thessaloniki, Thessaloniki, Greece;* [2]*Department of Theoretical Mechanics, University of Nottingham, Nottingham NG7 2RD, England*
(**author for correspondence*)

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Abstract. A thin-walled tube of finite length composed of a neo-Hookean material is sliding on a rough rigid rod under the action of forces distributed on its leading edge. A perturbation method is used to determine the stress and shape of the deformed tube to the second-order.

1. Introduction

The theory of large elastic deformations of membranes subject to normal surface tractions is well established [1]. For the solution of various physical problems it is necessary to extend the theory to the case where tangential as well as normal tractions are applied on one or both the surfaces of the membrane. For the derivation of such an extended theory from the three-dimensional equations of finite elasticity it is useful to have some information about the behaviour of a thin sheet under the action of shear surface forces. As a preliminary to the formulation of a general theory we consider the following special problem:

A thin, cylindrical elastic tube of circular cross-section, finite length L and inside radius R_0 in its undeformed state, slides slowly on a rod of radius $r_0 > R_0$, at constant speed, under the action of forces distributed on its leading edge. Its other edge, as well as its outside surface are free of applied forces. Moreover, we assume that the rod exerts on the inside surface of the tube a friction force which obeys the Coulomb friction law. The corresponding mathematical problem considered is as follows:

The undeformed body is a right circular tube composed of an elastic material, of length L, inside radius R_0 and thickness H. The radius of the outside surface will be denoted by $R_1 = R_0 + H$. The deformed body is a solid of revolution, its inside surface being a cylinder of radius r_0, and it is held in quasi-static equilibrium under the action of forces distributed along one of its edges and on its inside surface. The other edge, as well as the outside surface, are free from applied tractions. Sliding friction conditions apply at the inside surface.

We refer both the deformed and undeformed body to the same cartesian axes $OXYZ$, OZ being their common axis of symmetry, and assume that the material point of the deformed body with cylindrical coordinates (r, θ, z) had, in the undeformed state, the coordinates (R, Θ, Z) where

$$r = r(R, Z), \quad \theta = \Theta, \quad z = z(R, Z). \tag{1.1}$$

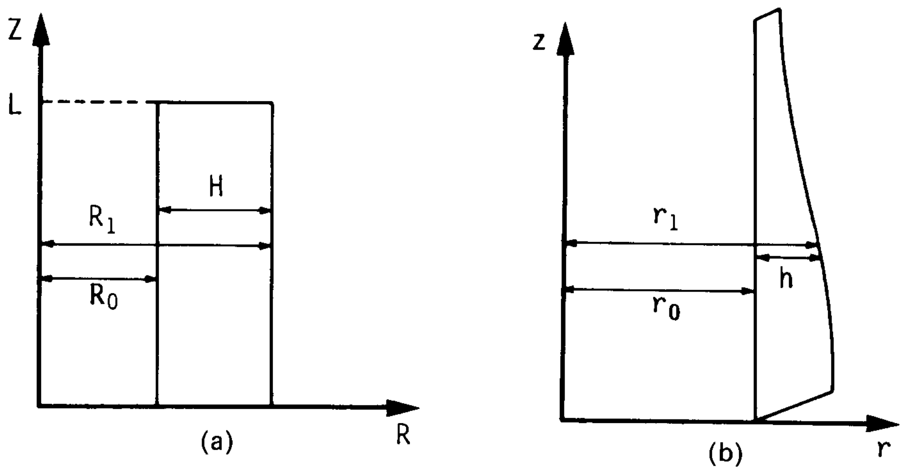

Fig. 1. Axial sections of the undeformed (a) and deformed (b) tube. Only the right half of the tube is shown.

Moreover, we assume that the undeformed body occupies the region $R_0 \leqq R \leqq R_1$, $0 \leqq Z \leqq L$, while the inside surface of the deformed body is the cylinder $r = r_0$, $0 \leqq z \leqq l$.

We make use of some standard results from the theory of Finite Elasticity. For a proof of formulae (1.2)–(1.7) the reader is referred to, for example, Spencer [2] the notation of which we follow whenever possible.

From (1.1) we easily obtain the deformation gradient tensor

$$\mathbf{F}^* = \begin{pmatrix} \partial r/\partial R & 0 & \partial r/\partial Z \\ 0 & r/R & 0 \\ \partial z/\partial R & 0 & \partial z/\partial Z \end{pmatrix} \tag{1.2}$$

and hence the incompressibility condition

$$\det \mathbf{F}^* = \frac{r}{R}\left(\frac{\partial r}{\partial R}\frac{\partial z}{\partial Z} - \frac{\partial r}{\partial Z}\frac{\partial z}{\partial R}\right) = 1, \tag{1.3}$$

and the left Cauchy-Green strain tensor

$$\mathbf{B}^* = \mathbf{F}^*(\mathbf{F}^*)^T = \begin{pmatrix} \left(\frac{\partial r}{\partial R}\right)^2 + \left(\frac{\partial r}{\partial Z}\right)^2 & 0 & \frac{\partial r}{\partial R}\frac{\partial z}{\partial R} + \frac{\partial r}{\partial Z}\frac{\partial z}{\partial Z} \\ 0 & \left(\frac{r}{R}\right)^2 & 0 \\ \frac{\partial r}{\partial R}\frac{\partial z}{\partial R} + \frac{\partial r}{\partial Z}\frac{\partial z}{\partial Z} & 0 & \left(\frac{\partial z}{\partial R}\right)^2 + \left(\frac{\partial z}{\partial Z}\right)^2 \end{pmatrix}. \tag{1.4}$$

We consider here the case of a neo-Hookean material which has stress–strain relations

$$\mathbf{T}^* = -p\mathbf{I} + 2C\mathbf{B}^*, \tag{1.5}$$

where $\mathbf{T}^*$ is the Cauchy stress referred to (r, θ, z) coordinates and p is an arbitrary pressure. This relation and (1.4) give

$$\mathbf{T}^* = 2C\begin{pmatrix} -\dfrac{p}{2C} + \left(\dfrac{\partial r}{\partial R}\right)^2 + \left(\dfrac{\partial z}{\partial R}\right)^2 & 0 & \dfrac{\partial r}{\partial R}\dfrac{\partial z}{\partial R} + \dfrac{\partial r}{\partial Z}\dfrac{\partial z}{\partial Z} \\ 0 & -\dfrac{p}{2C} + \left(\dfrac{r}{R}\right)^2 & 0 \\ \dfrac{\partial r}{\partial R}\dfrac{\partial z}{\partial R} + \dfrac{\partial r}{\partial Z}\dfrac{\partial z}{\partial Z} & 0 & -\dfrac{p}{2C} + \left(\dfrac{\partial z}{\partial R}\right)^2 + \left(\dfrac{\partial z}{\partial Z}\right)^2 \end{pmatrix}. \tag{1.6}$$

If we denote by

$$\tau = \mathbf{T}^*/2C$$

the non-dimensional stress tensor and assume zero body forces the equations of equilibrium can be expressed in the form

$$\begin{aligned} &\frac{\partial}{\partial R}(r\tau_{rr})\frac{\partial z}{\partial Z} - \frac{\partial}{\partial Z}(r\tau_{rr})\frac{\partial z}{\partial R} + r\frac{\partial(r, \tau_{rz})}{\partial(R, Z)} - \tau_{\vartheta\vartheta}\frac{\partial(r, z)}{\partial(R, Z)} = 0, \\ &\frac{\partial}{\partial \Theta}\left(\frac{p}{2C}\right) = 0, \\ &\frac{\partial}{\partial R}(r\tau_{rz})\frac{\partial z}{\partial Z} - \frac{\partial}{\partial Z}(r\tau_{rz})\frac{\partial z}{\partial R} + r\frac{\partial(r, \tau_{zz})}{\partial(R, Z)} = 0. \end{aligned} \tag{1.7}$$

where we have taken R, Θ, Z as the independent variables. In the specified configuration we also have the conditions

$$r(R_0, Z) = r_0, \quad z(R_0, 0) = 0. \tag{1.8}$$

The outside surface of the deformed body has parametric equations $r = r(R_1, Z)$, $z = z(R_1, Z)$, so that the direction ratios of the normal to this surface are $(\partial z/\partial Z, 0, -\partial r/\partial Z)_{R=R_1}$. The surface of the trailing edge has equations $r = r(R, 0)$, $z = z(R, 0)$ and the direction ratios of its normal are $(\partial z/\partial R, 0, -\partial r/\partial R)_{Z=0}$. Hence, because the surfaces $R = R_1$ and $Z = 0$ are free of external forces, we have, respectively, the conditions

$$\frac{\partial z}{\partial Z}\tau_{rr} - \frac{\partial r}{\partial Z}\tau_{zr} = 0, \quad \frac{\partial z}{\partial Z}\tau_{rz} - \frac{\partial r}{\partial Z}\tau_{zz} = 0 \quad \text{at} \quad R = R_1, \tag{1.9}$$

$$\frac{\partial z}{\partial R}\tau_{rr} - \frac{\partial r}{\partial R}\tau_{zr} = 0, \quad \frac{\partial z}{\partial R}\tau_{rz} - \frac{\partial r}{\partial R}\tau_{zz} = 0 \quad \text{at} \quad Z = 0. \tag{1.10}$$

Finally, the Coulomb friction condition at the inside surface of the deformed cylinder is

$$\tau_{rz} = -k\tau_{rr} \quad \text{at} \quad R = R_0, \tag{1.11}$$

where the coefficient of sliding friction k is assumed to be constant. We also note the obvious restriction

$$\tau_{rr} \leqq 0 \quad \text{at} \quad R = R_0. \tag{1.12}$$

2. Non-dimensional variables

We introduce the non-dimensional constants

$$\lambda = r_0/R_0, \quad \varepsilon = H/R_0, \tag{2.1}$$

and the non-dimensional variables

$$R = R_0 + Ht, \quad 0 \leqq t \leqq 1; \quad Z = R_0\zeta, \quad 0 \leqq \zeta \leqq L/R_0, \tag{2.2}$$

$$r(R, Z) = R_0 u(t, \zeta); \quad z(R, Z) = R_0 w(t, \zeta). \tag{2.3}$$

Then, the deformation gradient tensor, incompressibility condition, left Cauchy–Green deformation tensor and non-dimensional stress tensor take, respectively, the forms:

$$\mathbf{F}^* = \begin{pmatrix} \dfrac{1}{\varepsilon}\dfrac{\partial u}{\partial t} & 0 & \dfrac{\partial u}{\partial \zeta} \\ 0 & \dfrac{u}{1+\varepsilon t} & 0 \\ \dfrac{1}{\varepsilon}\dfrac{\partial w}{\partial t} & 0 & \dfrac{\partial w}{\partial \zeta} \end{pmatrix}, \tag{2.4}$$

$$\frac{u}{\varepsilon(1+\varepsilon t)} \cdot \frac{\partial(u, w)}{\partial(t, \zeta)} = 1, \tag{2.5}$$

$$\mathbf{B}^* = \begin{pmatrix} \dfrac{1}{\varepsilon^2}\left(\dfrac{\partial u}{\partial t}\right)^2 + \left(\dfrac{\partial u}{\partial \zeta}\right)^2 & 0 & \dfrac{1}{\varepsilon^2}\dfrac{\partial u}{\partial t}\dfrac{\partial w}{\partial t} + \dfrac{\partial u}{\partial \zeta}\dfrac{\partial w}{\partial \zeta} \\ 0 & \left(\dfrac{u}{1+\varepsilon t}\right)^2 & 0 \\ \dfrac{1}{\varepsilon^2}\dfrac{\partial u}{\partial t}\dfrac{\partial w}{\partial t} + \dfrac{\partial u}{\partial \zeta}\dfrac{\partial w}{\partial \zeta} & 0 & \dfrac{1}{\varepsilon^2}\left(\dfrac{\partial w}{\partial t}\right)^2 + \left(\dfrac{\partial w}{\partial \zeta}\right)^2 \end{pmatrix}, \tag{2.6}$$

$$\tau = \begin{pmatrix} -\frac{p}{2C} + \frac{1}{\varepsilon^2}\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial \zeta}\right)^2 & 0 & \frac{1}{\varepsilon^2}\frac{\partial u}{\partial t}\frac{\partial w}{\partial t} + \frac{\partial u}{\partial \zeta}\frac{\partial w}{\partial \zeta} \\ 0 & -\frac{p}{2C} + \left(\frac{u}{1+\varepsilon t}\right)^2 & 0 \\ \frac{1}{\varepsilon^2}\frac{\partial u}{\partial t}\frac{\partial w}{\partial t} + \frac{\partial u}{\partial \zeta}\frac{\partial w}{\partial \zeta} & 0 & -\frac{p}{2C} + \frac{1}{\varepsilon^2}\left(\frac{\partial w}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial \zeta}\right)^2 \end{pmatrix}, \tag{2.7}$$

while the equilibrium equations become

$$\begin{aligned} &\frac{\partial}{\partial t}(u\tau_{rr})\frac{\partial w}{\partial \zeta} - \frac{\partial}{\partial \zeta}(u\tau_{rr})\frac{\partial w}{\partial t} + u\frac{\partial(u, \tau_{rz})}{\partial(t, \zeta)} - \tau_{\vartheta\vartheta}\frac{\partial(u, w)}{\partial(t, \zeta)} = 0, \\ &\frac{\partial p}{\partial \theta} = 0, \\ &\frac{\partial}{\partial t}(u\tau_{rz})\frac{\partial w}{\partial \zeta} - \frac{\partial}{\partial \zeta}(u\tau_{rz})\frac{\partial w}{\partial t} + u\frac{\partial(u, \tau_{zz})}{\partial(t, \zeta)} = 0. \end{aligned} \tag{2.8}$$

Finally, the conditions (1.8) and the boundary conditions (1.9)–(1.11) take the forms

$$u(0, \zeta) = \lambda, \tag{2.9}$$

$$w(0, \zeta) = f(\zeta), \quad f(0) = 0, \tag{2.10}$$

$$\frac{\partial w}{\partial \zeta}\tau_{rr} - \frac{\partial u}{\partial \zeta}\tau_{zr} = 0, \quad \frac{\partial w}{\partial \zeta}\tau_{rz} - \frac{\partial u}{\partial \zeta}\tau_{zz} = 0 \quad \text{at} \quad t = 1, \tag{2.11}$$

$$\tau_{rz} = -k\tau_{rr} \quad \text{at} \quad t = 0, \tag{2.12}$$

$$\frac{\partial w}{\partial t}\tau_{rr} - \frac{\partial u}{\partial t}\tau_{rz} = 0, \quad \frac{\partial w}{\partial t}\tau_{rz} - \frac{\partial u}{\partial t}\tau_{zz} = 0 \quad \text{at} \quad \zeta = 0, \tag{2.13}$$

where, for convenience, we have introduced the notation $w(0, \zeta) = f(\zeta)$.

3. Series expansions. Preliminary result

For the problem, as stated, an analytic solution does not seem to be feasible. So, we try a regular perturbation solution based on the assumption that the thickness H of the undeformed body is much smaller than its inside radius R_0 and that all the functions to be determined can be expanded in power series of

$$\varepsilon = H/R_0 \ll 1. \tag{3.1}$$

If $\varepsilon \to 0$, R_0 remaining constant, then $r \to r_0$ and z tends to a function of $Z = R_0\zeta$ only. Hence we seek the expansion in series

$$u(t, \zeta) = \lambda + \varepsilon u_1(t, \zeta) + \dots, \quad w(t, \zeta) = f(\zeta) + \varepsilon w_1(t, \zeta) + \dots,$$

$$p/2C = p_0(t, \zeta) + \varepsilon p_1(t, \zeta) + \dots \ . \tag{3.2}$$

By substituting (3.2) in (2.7) we find that the components τ_{rz} and τ_{zz} of the non-dimensional stress tensor are

$$\tau_{rz} = \frac{\partial u_1}{\partial t}\frac{\partial w_1}{\partial t} + \varepsilon\left(\frac{\partial u_1}{\partial t}\frac{\partial w_2}{\partial t} + \frac{\partial u_2}{\partial t}\frac{\partial w_1}{\partial t} + \frac{\partial u_1}{\partial \zeta} f'\right) + O(\varepsilon^2),$$

$$\tau_{zz} = -p_0 + (f')^2 + \left(\frac{\partial w}{\partial t}\right)^2 + \varepsilon\left(-p_1 + 2\frac{\partial w_1}{\partial t}\frac{\partial w_2}{\partial t} + 2\frac{\partial w_1}{\partial \zeta} f'\right) + O(\varepsilon^2). \tag{3.3}$$

Similarly, a substitution of (3.2) in the boundary conditions (2.9) and (2.10) gives

$$u_1(0, \zeta) = 0, \tag{3.4}$$

$$f(0) = 0, \quad w_1(0, \zeta) = 0, \tag{3.5}$$

while from (3.2), (3.3) and $(2.11)_2$ we obtain, to the second-order in ε,

$$\frac{\partial u_1}{\partial t}\frac{\partial w_1}{\partial t} = 0 \quad \text{at} \quad t = 1. \tag{3.6}$$

From (3.2) and the incompressibility condition (2.5) we derive, to the zero-order,

$$\lambda \frac{\partial u_1}{\partial t} f'(\zeta) = 1,$$

which, together with the boundary condition (3.4), gives

$$u_1(t, \zeta) = \frac{t}{\lambda f'(\zeta)}. \tag{3.7}$$

The equilibrium equation $(2.8)_3$, (3.2) and (3.3) give, to the zero-order,

$$\frac{\partial}{\partial t}\left(\frac{\partial u_1}{\partial t}\frac{\partial w_1}{\partial t}\right) = 0.$$

Hence, because of (3.6),

$$\frac{\partial u_1}{\partial t}\frac{\partial w_1}{\partial t} = 0.$$

This result and (3.7) give $w_1 = w_1(\zeta)$. It follows, because of $(3.5)_2$, that $w_1 = 0$.

We note that the result $w_1 = 0$ is independent of the existence or not of shear forces on the inside surface. Hence it is valid for $k \geqq 0$.

4. Series expansions

Now that we have the result $w_1 = 0$, we start again with the expansions

$$u(t, \zeta) = \lambda + \varepsilon u_1(t, \zeta) + \dots, \quad w(t, \zeta) = f(\zeta) + \varepsilon^2 w_2(t, \zeta) + \dots,$$

$$p(t, \zeta)/2C = p_0(t, \zeta) + \varepsilon p_1(t, \zeta) + \dots . \tag{4.1}$$

The above series and the incompressibility condition (2.5) give, to the zero- and first-order, respectively,

$$\lambda \frac{\partial u_1}{\partial t} f' = 1, \quad \lambda \frac{\partial u_2}{\partial t} + (u_1 - \lambda t)\frac{\partial u_1}{\partial t} = 0. \tag{4.2}$$

If we use the notation

$$\boldsymbol{\tau} = \boldsymbol{\tau}^{(0)} + \varepsilon \boldsymbol{\tau}^{(1)} + \varepsilon^2 \boldsymbol{\tau}^{(2)} + \dots \tag{4.3}$$

for the non-dimensional stress tensor we derive, from (4.1) and (2.7), the formulae

$$\tau_{rr}^{(0)} = -p_0 + \left(\frac{\partial u_1}{\partial t}\right)^2, \quad \tau_{rr}^{(1)} = -p_1 + 2\frac{\partial u_1}{\partial t}\frac{\partial u_2}{\partial t},$$

$$\tau_{rz}^{(0)} = 0, \quad \tau_{rz}^{(1)} = \frac{\partial u_1}{\partial t}\frac{\partial w_2}{\partial t} + \frac{\partial u_1}{\partial \zeta} f',$$

$$\tau_{rz}^{(2)} = \frac{\partial u_2}{\partial \zeta} f' + \frac{\partial u_2}{\partial t}\frac{\partial w_2}{\partial t} + \frac{\partial u_1}{\partial t}\frac{\partial w_3}{\partial t}, \tag{4.4}$$

$$\tau_{\vartheta\vartheta}^{(0)} = -p_0 + \lambda^2, \quad \tau_{\vartheta\vartheta}^{(1)} = -p_1 + 2\lambda(u_1 - \lambda t),$$

$$\tau_{zz}^{(0)} = -p_0 + (f')^2, \quad \tau_{zz}^{(1)} = -p_1,$$

$$\tau_{r\vartheta} = \tau_{z\vartheta} = 0,$$

while (4.1) and (2.8) give the zero- and first-order equilibrium equations

$$\frac{\partial \tau_{rr}^{(0)}}{\partial t} = 0, \quad p_0 = p_0(t, \zeta),$$

$$\frac{\partial \tau_{rz}^{(1)}}{\partial t} f' + \frac{\partial(u_1, \tau_{zz}^{(0)})}{\partial(t, \zeta)} = 0, \tag{4.5}$$

$$\frac{\partial}{\partial t}(u_1 \tau_{rr}^{(0)} + \lambda \tau_{rr}^{(1)}) - \tau_{\vartheta\vartheta}^{(0)} \frac{\partial u_1}{\partial t} = 0, \quad p_1 = p_1(t, \zeta),$$

$$f' \frac{\partial}{\partial t}(\lambda \tau_{rz}^{(2)} + u_1 \tau_{rz}^{(1)}) + u_1 \frac{\partial(u_1, \tau_{zz}^{(0)})}{\partial(t, \zeta)} + \lambda \left[\frac{\partial(u_1, \tau_{zz}^{(1)})}{\partial(t, \zeta)} + \frac{\partial(u_2, \tau_{zz}^{(0)})}{\partial(t, \zeta)} \right] = 0. \tag{4.6}$$

In the same way we obtain from (2.9)–(2.10) the conditions

$$u_1(0, \zeta) = u_2(0, \zeta) = \ldots = 0,$$

$$f(0) = 0, \quad w_2(0, \zeta) = w_3(0, \zeta) = \ldots = 0, \tag{4.7}$$

and, from (2.11)–(2.13), the zero- and first-order boundary conditions

$$\tau_{rr}^{(0)} = 0, \quad f'\tau_{rz}^{(1)} - \frac{\partial u_1}{\partial \zeta} \tau_{zz}^{(0)} = 0 \quad \text{at} \quad t = 1,$$

$$k\tau_{rr}^{(0)} = 0 \quad \text{at} \quad t = 0,$$

$$\frac{\partial u_1}{\partial t} \tau_{rz}^{(1)} - \frac{\partial w_2}{\partial t} \tau_{rr}^{(0)} = 0, \quad \frac{\partial u_1}{\partial t} \tau_{zz}^{(0)} = 0 \quad \text{at} \quad \zeta = 0; \tag{4.8}$$

$$\tau_{rr}^{(1)} = 0, \quad f'\tau_{rz}^{(2)} - \frac{\partial u_1}{\partial \zeta} \tau_{zz}^{(1)} - \frac{\partial u_2}{\partial \zeta} \tau_{zz}^{(0)} = 0 \quad \text{at} \quad t = 1,$$

$$\tau_{rz}^{(1)} + k\tau_{rr}^{(1)} = 0 \quad \text{at} \quad t = 0,$$

$$\frac{\partial w_2}{\partial t} \tau_{rr}^{(1)} + \frac{\partial w_3}{\partial t} \tau_{rr}^{(0)} - \frac{\partial u_1}{\partial t} \tau_{rz}^{(2)} - \frac{\partial u_2}{\partial t} \tau_{rz}^{(1)} = 0 \quad \text{at} \quad \zeta = 0,$$

$$\frac{\partial u_2}{\partial t} \tau_{zz}^{(0)} + \frac{\partial u_1}{\partial t} \tau_{zz}^{(1)} = 0 \quad \text{at} \quad \zeta = 0. \tag{4.9}$$

5. Solutions

From the zero-order incompressibility condition $(4.2)_1$ and $(4.7)_1$, we derive

$$u_1 = \frac{t}{\lambda f'(\zeta)}. \tag{5.1}$$

By substituting this result in the first-order incompressibility condition $(4.2)_2$ and using $(4.7)_1$ we obtain

$$u_2 = \frac{\lambda^2 f' - 1}{2\lambda^3 (f')^2} t^2. \tag{5.2}$$

The zero-order equilibrium equation $(4.5)_1$ and the boundary condition $(4.8)_1$ give

$$\tau_{rr}^{(0)} \equiv 0 \tag{5.3}$$

from which, because of $(4.4)_1$ and $(5.1)_1$ it follows that

$$p_0 = (\lambda f')^{-2}. \tag{5.4}$$

From the above results, (4.4) and the first-order equilibrium equation $(4.6)_1$ we obtain

$$\frac{\partial \tau_{rr}^{(1)}}{\partial t} = \frac{1}{\lambda}(\lambda^2 - p_0)\frac{\partial u_1}{\partial t}. \tag{5.5}$$

Hence, because of the boundary condition $(4.9)_1$,

$$\tau_{rr}^{(1)} = \frac{t-1}{\lambda}(\lambda^2 - p_0)\frac{\partial u_1}{\partial t}. \tag{5.6}$$

The substitution of $(4.4)_2$ in this expression for $\tau_{rr}^{(1)}$ gives

$$p_1 = -\frac{(\lambda^2 f' - 1)^2}{\lambda^4 (f')^3} t + \frac{\lambda^4 (f')^2 - 1}{\lambda^4 (f')^3}. \tag{5.7}$$

If we substitute the expressions for $\tau_{zz}^{(0)}$ and $\tau_{rz}^{(1)}$ from (4.4) in the third second-order equation $(4.5)_3$ and use the above expressions for p_0 and u_1 we obtain

$$\frac{\partial^2 w_2}{\partial t^2} = -\frac{\lambda^2 (f')^4 + 2}{\lambda^2 (f')^4} f'' \tag{5.8}$$

from which, because of $(4.7)_3$,

$$w_2 = -\frac{\lambda^2 (f')^4 + 2}{2\lambda^2 (f')^4} f'' t^2 + tA(\zeta). \tag{5.9}$$

The integration function $A(\zeta)$ is determined from the second zero-order boundary condition $(4.8)_2$. A substitution in (5.9) gives

$$w_2 = \frac{2t[\lambda^2(f')^4 + 3] - t^2[\lambda^2(f')^4 + 2]}{2\lambda^2(f')^4} f''. \tag{5.10}$$

Finally, from the first-order friction boundary condition $(4.9)_3$ and the expressions for $\tau_{rz}^{(1)}$ and $\tau_{rr}^{(1)}$ we derive

$$f'' = \frac{k[\lambda^4(f')^2 - 1](f')^2}{\lambda[\lambda^2(f')^4 + 3]}. \tag{5.11}$$

The initial conditions which determine the required solution of this equation are

$$f(0) = 0, \quad f'(0) = 1/\sqrt{\lambda}. \tag{5.12}$$

The first of (5.12) is $(4.7)_2$ and the second satisfies the boundary condition $(4.8)_5$ as well as the requirement that $f(\zeta)$ is an increasing function of ζ for small ζ. Equation (5.11) is a first-order differential equation for $f'(\zeta)$ which can be integrated to give

$$\begin{aligned} \zeta &= Q[f'(\zeta), k, \lambda] \\ &= \frac{1}{k}\left[\frac{(f')^2 + 3\lambda^2}{\lambda f'} + \frac{1 + 3\lambda^6}{2\lambda^3}\log\frac{\lambda^2 f' - 1}{\lambda^2 f' + 1} - \frac{1 + 3\lambda^3}{\lambda\sqrt{\lambda}}\right. \\ &\quad \left. + \frac{1 + 3\lambda^6}{2\lambda^3}\log\frac{\lambda^2 + \sqrt{\lambda}}{\lambda^2 - \sqrt{\lambda}}\right], \quad f(0) = 0, \end{aligned} \tag{5.13}$$

where the boundary condition $(5.12)_2$ has been satisfied. The change of variable $f'(\zeta) = \tau$ in (5.13) gives, in the usual way, the parametric representation of the solution to (5.11):

$$\zeta = Q(\tau, k, \lambda), \quad f(\tau) = \int_{1/\sqrt{\lambda}}^{\tau} \xi \frac{\mathrm{d}Q(\xi, k, \lambda)}{\mathrm{d}\xi}\,\mathrm{d}\xi, \quad \tau \geqq 1/\sqrt{\lambda}. \tag{5.14}$$

It follows that the solution to (5.11) is, in parametric form

$$\left.\begin{aligned} \zeta(\tau) &= \frac{1}{k}\left[\frac{\tau^2 + 3\lambda^2}{\lambda\tau} + \frac{1 + 3\lambda^6}{2\lambda^3}\log\frac{\lambda^2\tau - 1}{\lambda^2\tau + 1} - \frac{1 + 3\lambda^3}{\lambda\sqrt{\lambda}}\right. \\ &\quad \left. + \frac{1 + 3\lambda^6}{2\lambda^3}\log\frac{\lambda^2 + \sqrt{\lambda}}{\lambda^2 - \sqrt{\lambda}}\right], \\ f(\tau) &= \frac{1}{k}\left[\frac{\lambda\tau^2 - 1}{2\lambda^2} - 3\lambda\log\tau\sqrt{\lambda} + \frac{1 + 3\lambda^6}{2\lambda^5}\log\frac{\lambda^4\tau^2 - 1}{\lambda^3 - 1}\right] \end{aligned}\right\} \tau \geqq 1/\sqrt{\lambda}. \tag{5.15}$$

From (5.11) we see that $f'(\zeta)$ is an increasing function of ζ if $f' > \lambda^{-2}$. We also have $f'(0) = 1/\sqrt{\lambda} > 1/\lambda^2$ from $(5.12)_2$. Hence $f'(\zeta)$ is a positive increasing function of ζ and

$f'(\zeta) \geqq 1/\sqrt{\lambda} > \lambda^{-2}$. The justifies the condition $\tau \geqq 1/\sqrt{\lambda}$ in (5.14) and (5.15). It also follows that $f(\zeta)$ is increasing and, from (5.1), (5.4) and (5.6), that $\tau_{rr}^{(1)}(t, \zeta) < 0$ for $0 \leqq t < 1$, $\tau_{rr}^{(1)}(1, \zeta) = 0$, as was to be expected from physical considerations.

The quantities u_1, u_2, p_0, p_1 and w_2 are determined, respectively, by (5.1), (5.2), (5.4) and (5.9) in terms of the solution $f(\zeta)$ given by (5.15). They satisfy, up to the first-orders in ε, the equations of equilibrium except the first-order equation $(4.6)_3$ which involves $\tau_{rz}^{(2)}$ and, hence, w_3. They also satisfy, up to the first-order, all the boundary conditions on the inside and outside surface except $(4.9)_2$ which, together with the equation $(4.6)_3$ can be used to determine $\tau_{rz}^{(2)}$ if necessary. The assumption that no surface tractions are applied on the edge $z = 0$ is satisfied to the zero-order.

A noteworthy feature of the solution is that the non-dimensional stress components τ_{rr} and τ_{rz} are of order ε. This observation may provide the basis for a more general theory of axisymmetric membranes with tangential tractions. We shall show below that, nevertheless, the effect of tangential tractions on the deformation is very substantial.

The solution given above is valid for $k > 0$. For comparison we now consider the frictionless case $k = 0$. As noted at the end of §3 the result $w_1 = 0$ is still valid and if we tentatively assume the same expansions (4.1) we again obtain the same formulae (4.2)–(4.9) and (5.1)–(5.10) where $(4.8)_3$ is identically satisfied and $(4.9)_3$ is replaced by

$$\tau_{rz}^{(1)} = 0 \quad \text{at} \quad t = 0. \tag{5.16}$$

From $(4.4)_4$, the values of u_1, w_2 already determined and the above boundary condition we derive $f''(\zeta) = 0$ from which

$$f(\zeta) = A\zeta, \tag{5.17}$$

where the condition $f(0) = 0$ has been satisfied and A is a constant to be determined. From $(4.4)_8$, (5.4), (5.17) and the zero-order boundary condition $(4.8)_5$ we obtain $A = 1/\sqrt{\lambda}$. It follows that

$$\left.\begin{aligned} u_1 &= t/\sqrt{\lambda}, \quad u_2 = \frac{\lambda\sqrt{\lambda} - 1}{2\lambda^2}\,t^2, \qquad w_1 = w_2 = 0, \\ p_0 &= 1/\lambda, \quad p_1 = -\frac{(\lambda\sqrt{\lambda} - 1)^2}{\lambda^2\sqrt{\lambda}}\,t + \frac{\lambda^3 - 1}{\lambda^2\sqrt{\lambda}}, \quad f(\zeta) = \zeta/\sqrt{\lambda}. \end{aligned}\right\} k = 0. \tag{5.18}$$

The above solution satisfies the equilibrium equations (4.5)–(4.6) except the first-order equation $(4.6)_3$ which involves w_3. It also satisfies the boundary conditions on the inside and outside surface to the first-order as well as the assumption that no tractions are applied on the edge $\zeta = 0$ to the zero-order.

6. Numerical results

Some numerical results, in a non-dimensional form, are given in this paragraph. The undeformed solid is assumed to be of length $L = 6R_0$, i.e., $0 \leqq \zeta \leqq 6$.

The non-dimensional length of the deformed tube is, to the second-order, $w(t, 6) = f(6) + \varepsilon^2 w_2(t, 6)$, where $w_2(0, \zeta) = 0$. It follows that the non-dimensional length of the contact surface is $f(6)$ which can be obtained from (5.15), or (5.18) in the particular case $k = 0$. Otherwise, since $f'(\zeta)$ is an increasing function of ζ, one can use (5.13) and the formula

$$f(\zeta) = \zeta f'(\zeta) - \int_{1/\sqrt{\lambda}}^{f'(\zeta)} Q(\xi, k, \lambda)\, d\xi. \tag{6.1}$$

In Fig. 2, $f(6)$ is plotted against λ, $1 \leqq \lambda \leqq 2$, for nine values of the friction coefficient k, $0 \leqq k_i \leqq 0.8$. We note that a typical value of the coefficient of kinetic friction of rubber on metal is $k = 0.3$. For constant λ the deformed length is an increasing function of k and, for constant $k \geqq 0.1$, an increasing function of λ, the steepest increase being for values of λ close to 1. For small values of k, say $0 \leqq k \leqq 0.1$, f appears to be a decreasing function of λ.

It is easily seen that the total axial traction F applied on the surface ζ = constant of

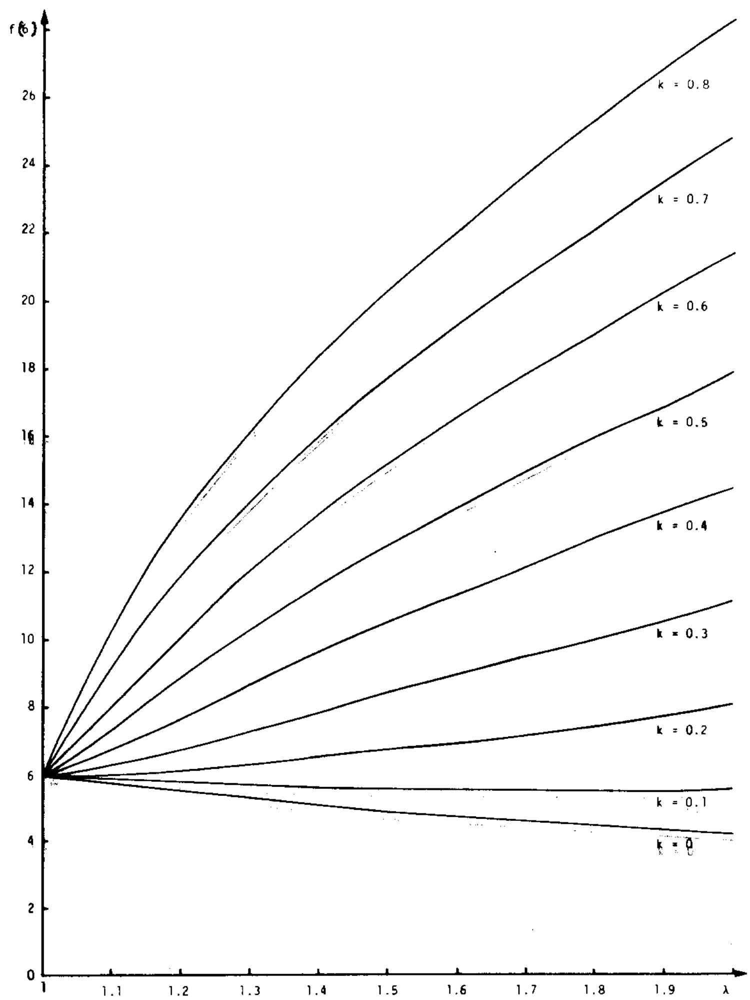

Fig. 2. The dependence of $f(6)$ on λ and k.

the deformed body is given by the formula

$$\frac{F}{4\pi C R_0^2} = \varepsilon F_1 + \varepsilon^2 F_2, \quad F_1 = \frac{\lambda^2 (f')^4 - 1}{\lambda^2 (f')^3},$$

$$F_2 = \frac{1}{2(\lambda f')^4}\{(\lambda^2 f' - 1)^2 + \lambda^2 f'[\lambda^2 (f')^4 - 1] + 2[1 - \lambda^4 (f')^2]\}. \tag{6.2}$$

For $\zeta = 6$ the above formula gives the external force applied on the leading edge of the sliding tube. In Fig. 3, the coefficient F_1 in the first-order approximation εF_1 is plotted against λ for eight different values of k. It is seen that F_1 is an increasing function of k for each value of λ and, for constant k, it is an increasing function of λ, the increase being steeper for values of λ close to 1.

To second-order, the material lines which originally were straight lines in the $\mathbf{e}_r$ direction deform to parabolas with slope

$$\partial z/\partial r = \varepsilon g_1(t, \zeta) + \varepsilon^2 g_2(t, \zeta),$$

$$g_1(t, \zeta) = \frac{k\{\lambda^2 (f')^4 + 3 - t[\lambda^2 (f')^4 + 2]\}[\lambda^4 (f')^2 - 1]}{\lambda^2 f'[\lambda^2 (f')^4 + 3]}, \tag{6.3}$$

$$g_2(t, \zeta) = -g_1(t, \zeta)\frac{(\lambda^2 f' - 1)t}{\lambda^2 f'},$$

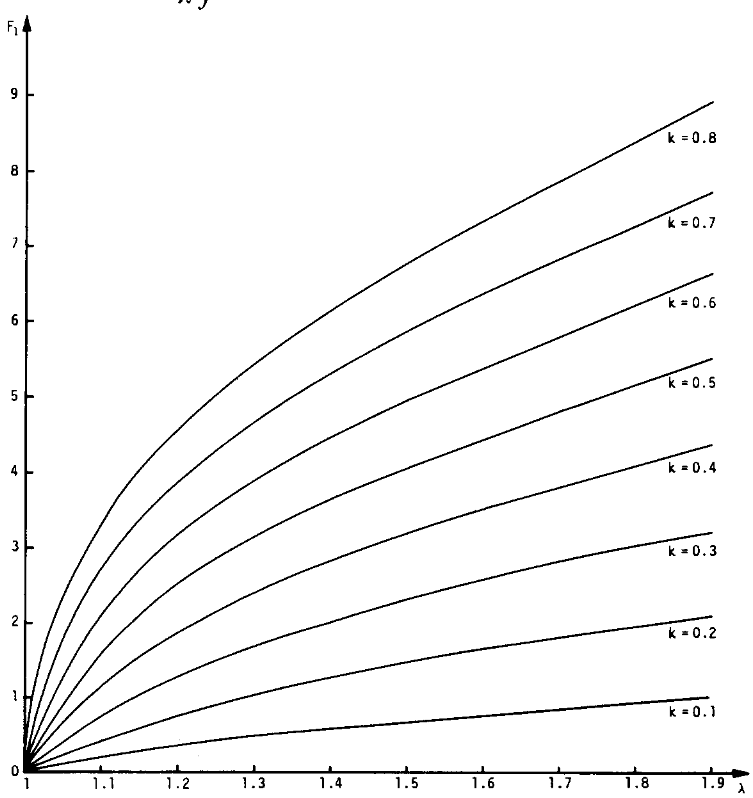

Fig. 3. The dependence of F_1 on λ and k.

which can easily be calculated from (2.3), (5.1), (5.2) and (5.10). We observe that, to the second-order, the slope is proportional to k. As already noted $f'(\zeta)$ is an increasing function of ζ and $f'(0) = 1/\sqrt{\lambda} > 1/\lambda^2$. It follows that the slope and the coefficients $g_1(t, \zeta)$, $g_2(t, \zeta)$ in (6.3) are decreasing functions of t, where $g_1 > 0$ and $g_2 < 0$. For the coefficient $g_1(t, \zeta)$ of the leading term we have

$$g_1(0, \zeta)/k = \frac{\lambda^4(f')^2 - 1}{\lambda^2 f'} \geqq g_1(t, \zeta)/k \geqq \frac{\lambda^4(f')^2 - 1}{\lambda^2 f'[\lambda^2(f')^4 + 3]} = g_1(1, \zeta). \tag{6.4}$$

The extreme values $g_1(0, \zeta)/k$ and $g_1(1, \zeta)/k$ of $g_1(t, \zeta)/k$ are plotted against ζ in Fig. 4 for four different values of λ. The maximum value $g_1(0, \zeta)/k$, attained at the surface of contact, appears to be an increasing function of ζ and for constant ζ, an increasing function of λ. The minimum value $g_1(1, \zeta)/k$, obtained at the free surface is, for constant ζ, an increasing

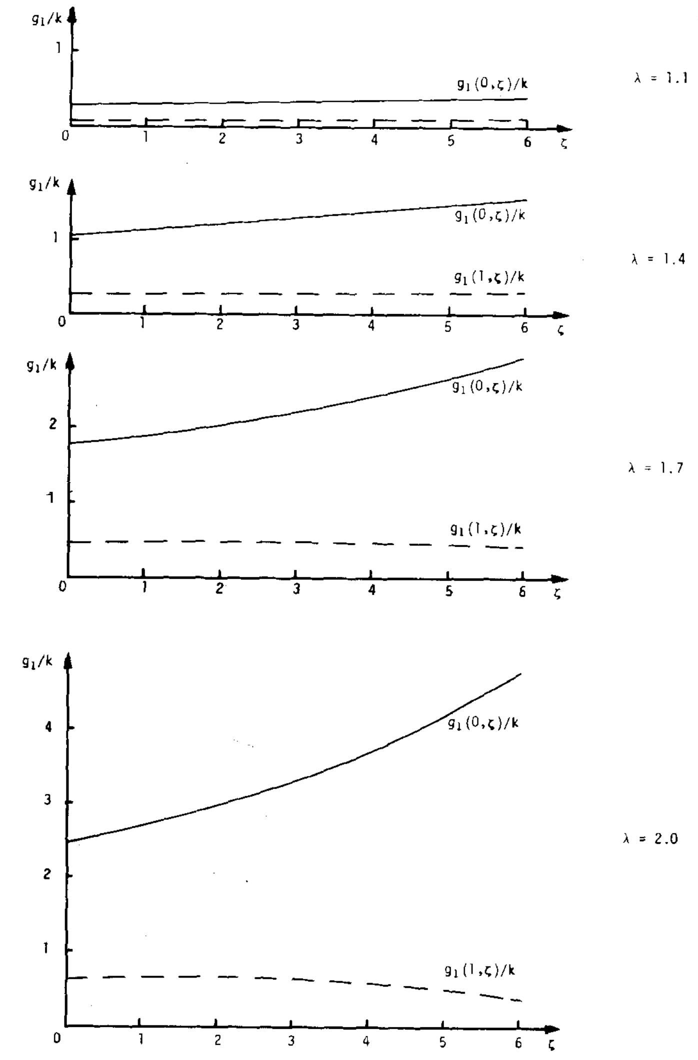

Fig. 4. The dependence of $g_1(0, \zeta)/k$ and $g_1(1, \zeta)/k$ on ζ for $\lambda = 1.1, 1.4, 1.7, 2.0$.

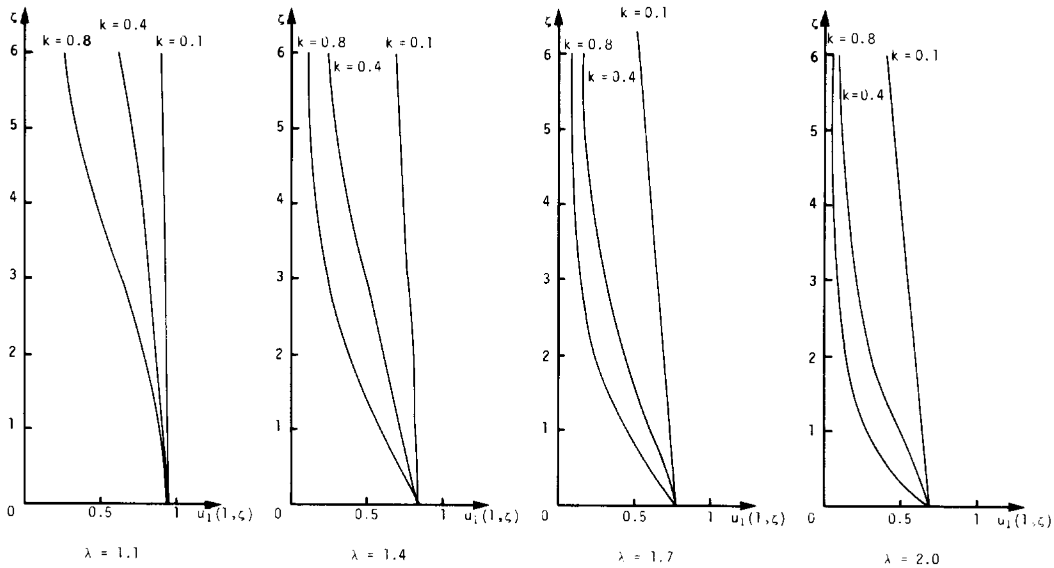

Fig. 5. The dependence of $u_1(1, \zeta)$ on ζ for $\lambda = 1.1, 1.4, 1.7, 2.0$ and $k = 0.1, 0.4, 0.8$.

function of λ. Upper bounds for the slope at $t = 0$ and $t = 1$ are easily derived:

$$\partial z/\partial r|_{t=0} \leqq \varepsilon k \lambda^2 f'(L/R_0), \quad \partial z/\partial r|_{t=1} \leqq \varepsilon k \lambda \sqrt{\lambda}. \tag{6.5}$$

We note that they are proportional to k and that the upper bound given for the slope at $t = 1$ is independent of the length of the tube.

The non-dimensional "thickness" of the deformed tube is given by

$$u(1, \zeta) - \lambda = \varepsilon u_1(1, \zeta) + \varepsilon^2 u_2(1, \zeta). \tag{6.6}$$

Figure 5 shows the variation of $u_1(1, \zeta)$ in terms of ζ for $\lambda = 1.1, 1.4, 1.7, 2.0$ and $k = 0.1, 0.4, 0.8$. $u_1(1, \zeta)$ is a decreasing function of ζ and for a given value of ζ, $\zeta > 0$, a decreasing function of both λ and k. We note that the axial stretch $f'(\zeta)$ can be calculated from Fig. 5 since $f'(\zeta) = 1/\lambda u_1(1, \zeta)$ from (5.1).

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